

ON THE ARITHMETIC FUNDAMENTAL GROUPS

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ABSTRACT. In this paper we will define a qc fundamental group for an arithmetic scheme by quasi-galois closed covers. Then we will give a computation for such a group and will prove that the étale fundamental group of an arithmetic scheme is a normal subgroup in our qc fundamental group, which make up the main theorem of the paper. Hence, our group gives us a prior estimate of the étale fundamental group. The quotient group reflects the topological properties of the scheme.

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INTRODUCTION

One has been used the related data of arithmetic varieties X/Y to describe a given Galois extension E/F for many years (for example, see [5, 10, 11, 15, 16, 17, 19, 20]). It has been seen that there is a nice relationship between the Galois group $Gal(E/F)$ and the automorphism group $Aut(X/Y)$ for the case that E/F are canonically the function fields $k(X)/k(Y)$.

Here, one says that the arithmetic varieties X/Y are a *geometric model* for the Galois extensions E/F if the Galois group $Gal(E/F)$ is isomorphic to the automorphism group $Aut(X/Y)$ (for example, see [8, 15, 17, 18]).

Among the whole of the invariants on arithmetic schemes, it has been seen that the étale fundamental groups are a very important tool for one to obtain class fields since these groups encode the whole of the information of class fields of the number fields or function fields. So, it is a natural task how to compute the étale fundamental groups.

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In this paper we will define a qc fundamental groups for an arithmetic scheme by quasi-galois closed covers in a manner similar to an étale fundamental group of the scheme. Here, as behave like Galois extensions, the quasi-galois closed covers have many desired properties of the arithmetic schemes (for example, see [2, 3]).

In deed, quasi-galois closed covers can be regarded as a generalization of the pseudo-galois covers of arithmetic varieties in the sense of Suslin-Voevodsky (see [17, 18]).

In *Theorem 2.2*, the main theorem of the paper, we will give a computation for a qc fundamental group of an arithmetic scheme; we will also prove that the étale fundamental group of the scheme is a normal subgroup in our qc fundamental group. Hence, by our groups we will obtain a prior estimate of the étale fundamental groups.

On the other hand, their quotient group reflect the topological properties of the scheme. It will be seen that an arithmetic scheme has no finite branched cover if and only if the quotient group is trivial.

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1. DEFINITION AND NOTATION

1.1. Convention. In this paper, an **arithmetic variety** is an integral scheme surjectively over $\text{Spec}(\mathbb{Z})$ of finite type; an **integral variety** is an integral scheme surjectively over $\text{Spec}(\mathbb{Z})$. Let $k(X)$ denote the function field of an integral scheme X .

For an integral domains D , we will let $Fr(D)$ denote the field of fractions on D . In particular, the field $Fr(D)$ will always assumed to be contained in Ω if D is a subring of a field Ω .

For a field K , we let K^{al} denote the algebraically closed closure of K . Define K^{un} to be the union of all the finite unramified subextensions $E (\subseteq K^{al})$ over K .

For an arithmetic variety X , let $\pi_1^{et}(X; s)$ denote the étale fundamental group of X for a given geometric point s over X (for detail see [6, 8, 13]).

1.2. Affine covering with values in a field. For convenience, let us fix notation and recall some definitions in the paper (for details, see [1, 2, 3]). Fixed a scheme (X, \mathcal{O}_X) .

An **affine covering** of (X, \mathcal{O}_X) is a family $\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ such that ϕ_α is an isomorphism from an open set U_α of X onto the spectrum $\text{Spec} A_\alpha$ of a commutative ring A_α for any $\alpha \in \Delta$. Each

$(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ is called a **local chart**. Moreover, \mathcal{C}_X is said to be **reduced** if $U_\alpha \neq U_\beta$ holds for any $\alpha \neq \beta$ in Δ .

In particular, an **affine patching** of (X, \mathcal{O}_X) is an affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of (X, \mathcal{O}_X) such that ϕ_α is the identity map on $U_\alpha = \text{Spec} A_\alpha$ for each $\alpha \in \Delta$. Evidently, an affine patching is reduced.

Let **Comm** be the category of commutative rings with identity. Fixed a subcategory **Comm**₀ of **Comm**. An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of (X, \mathcal{O}_X) is said to be **with values** in **Comm**₀ if for each $\alpha \in \Delta$ there are $\mathcal{O}_X(U_\alpha) = A_\alpha$ and $U_\alpha = \text{Spec}(A_\alpha)$, where A_α is a ring contained in **Comm**₀.

Let Ω be a field and let **Comm**(Ω) be the category consisting of the subrings of Ω and their isomorphisms. An affine covering \mathcal{C}_X of (X, \mathcal{O}_X) with values in **Comm**(Ω) is said to be **with values in the field Ω** .

Let \mathcal{O}_X and \mathcal{O}'_X be two structure sheaves on the underlying space of an integral scheme X . The two integral schemes (X, \mathcal{O}_X) and (X, \mathcal{O}'_X) are said to be **essentially equal** provided that for any open set U in X , we have

$$U \text{ is affine open in } (X, \mathcal{O}_X) \iff \text{so is } U \text{ in } (X, \mathcal{O}'_X)$$

and in such a case, $D_1 = D_2$ holds or there is $Fr(D_1) = Fr(D_2)$ such that for any nonzero $x \in Fr(D_1)$, either

$$x \in D_1 \bigcap D_2$$

or

$$x \in D_1 \setminus D_2 \iff x^{-1} \in D_2 \setminus D_1$$

holds, where $D_1 = \mathcal{O}_X(U)$ and $D_2 = \mathcal{O}'_X(U)$.

Two schemes (X, \mathcal{O}_X) and (Z, \mathcal{O}_Z) are said to be **essentially equal** if the underlying spaces of X and Z are equal and the schemes (X, \mathcal{O}_X) and (X, \mathcal{O}_Z) are essentially equal.

1.3. Galois extensions. Let K be a finitely generated extension of a field k . Here K is not necessarily algebraic over k . As usual, $Gal(K/k)$ denotes the Galois group of K over k .

The field K is said to be a **Galois extension** of k if k is the fixed subfield of the Galois group $Gal(K/k)$ in K .

1.4. Quasi-galois closed. Let X and Y be two integral varieties and let $f : X \rightarrow Y$ be a surjective morphism. Denote by $Aut(X/Y)$ the group of automorphisms of X over Y .

By a **conjugate** Z of X over Y , we will understand an integral variety Z that is isomorphic to X over Y .

Definition 1.1. X is said to be **quasi-galois closed** over Y by f if there is an algebraically closed field Ω and a reduced affine covering \mathcal{C}_X of X with values in Ω such that for any conjugate Z of X over Y the two conditions are satisfied:

- (X, \mathcal{O}_X) and (Z, \mathcal{O}_Z) are essentially equal if Z has a reduced affine covering with values in Ω .
- $\mathcal{C}_Z \subseteq \mathcal{C}_X$ holds if \mathcal{C}_Z is a reduced affine covering of Z with values in Ω .

Remark 1.2. It is seen that a quasi-galois closed variety X has only one conjugate in the algebraically closed field $\Omega = k(X)^{al}$. That is, let Z and Z' be conjugates of X . Then Z and Z' must be essentially equal if Z and Z' both have reduced affine coverings with values in Ω .

2. STATEMENT OF THE MAIN THEOREM

2.1. Definition for qc fundamental group. Let X be an arithmetic variety. Fixed an algebraically closed field Ω such that the function field $k(X)$ is contained in Ω . Here, Ω is not necessarily algebraic over $k(X)$.

Define $X_{qc}[\Omega]$ to be the set of arithmetic varieties Z satisfying the following conditions: (i) Z has a reduced affine covering with values in Ω ; (ii) there is a surjective morphism $f : Z \rightarrow X$ of finite type such that Z is quasi-galois closed over X .

Set a partial order \leq in the set $X_{qc}[\Omega]$ in such a manner:

Take any $Z_1, Z_2 \in X_{qc}[\Omega]$, we say

$$Z_1 \leq Z_2$$

if there is a surjective morphism $\varphi : Z_2 \rightarrow Z_1$ of finite type such that Z_2 is quasi-galois closed over Z_1 .

By *Lemmas 3.6, 3.8-10* below, it is seen that $X_{qc}[\Omega]$ is a directed set and

$$\{Aut(Z/X) : Z \in X_{qc}[\Omega]\}$$

is an inverse system of groups. Hence, we have the following definition.

Definition 2.1. Let X be an arithmetic variety and let Ω be an algebraically closed field $\Omega \supseteq k(X)$. The inverse limit

$$\pi_1^{qc}(X; \Omega) \triangleq \varprojlim_{Z \in X_{qc}[\Omega]} Aut(Z/X)$$

of the inverse system $\{Aut(Z/X) : Z \in X_{qc}[\Omega]\}$ of groups is said to be the **qc fundamental group** of the scheme X with coefficient in Ω .

2.2. Statement of the main theorem. The following is the main theorems of the paper.

Theorem 2.2. *Let X be an arithmetic variety. Take any algebraically closed field Ω such that $\Omega \supseteq k(X)$. Then we have the following statements.*

(i) *There is a group isomorphism*

$$\pi_1^{qc}(X; \Omega) \cong \text{Gal}(\Omega/k(X)).$$

(ii) *Take any geometric point s of X over Ω . Then there is a group isomorphism*

$$\pi_1^{et}(X; s) \cong \pi_1^{qc}(X; \Omega)_{et}$$

where $\pi_1^{qc}(X; \Omega)_{et}$ is a subgroup of $\pi_1^{qc}(X; \Omega)$. Moreover, $\pi_1^{qc}(X; \Omega)_{et}$ is a normal subgroup of $\pi_1^{qc}(X; \Omega)$.

Remark 2.3. Let X be an arithmetic variety. Put

$$\pi_1^{qc}(X) = \pi_1^{qc}(X; k(X)^{al}).$$

Then there is a group isomorphism

$$\pi_1^{qc}(X) \cong \text{Gal}(k(X)^{al}/k(X)).$$

Definition 2.4. Let X be an arithmetic variety. The quotient group

$$\pi_1^{br}(X) = \pi_1^{qc}(X; k(X)^{al}) / \pi_1^{qc}(X; k(X)^{al})_{et}$$

is said to be the **branched group** of the arithmetic variety X .

The branched group $\pi_1^{br}(X)$ can reflect the topological properties of the scheme X , especially the properties of the associated complex space X^{an} of X , for example, the branched covers of X^{an} . In fact, we have the following corollary.

Corollary 2.5. Let X be an arithmetic variety. Then we have

$$\pi_1^{br}(X) = \{0\}$$

if and only if X has no finite branched cover.

Proof. Trivial. □

3. PROOF OF THE MAIN THEOREM

3.1. Recalling definitions and basic facts. Let us recall some definitions and basic facts about quasi-galois closed schemes (see [2, 3, 4]).

Let K be a finitely generated extension of a field k . Here K is not necessarily algebraic over k .

Definition 3.1. (see [2]) K is **quasi-galois** over k if each irreducible polynomial $f(X) \in F[X]$ that has a root in K factors completely in $K[X]$ into linear factors for any intermediate field $k \subseteq F \subseteq K$.

We call the elements $t_1, t_2, \dots, t_n \in K \setminus k$ a (r, n) -**nice basis** of K over k if the conditions are satisfied:

- (i) $L = K(t_1, t_2, \dots, t_n)$;
- (ii) t_1, t_2, \dots, t_r constitute a transcendental basis of L over K ;
- (iii) $t_{r+1}, t_{r+2}, \dots, t_n$ are linearly independent over $K(t_1, t_2, \dots, t_r)$, where $0 \leq r \leq n$.

Definition 3.2. (see [2]) Let $D \subseteq D_1 \cap D_2$ be three integral domains. The ring D_1 is said to be **quasi-galois** over D if the field $Fr(D_1)$ is a quasi-galois extension of $Fr(D)$.

Definition 3.3. (see [2]) The ring D_1 is a **conjugation** of D_2 over D if there is a (r, n) -nice k -basis w_1, w_2, \dots, w_n of the field $Fr(D_1)$ and an F -isomorphism $\tau_{(r,n)} : Fr(D_1) \rightarrow Fr(D_2)$ of fields such that

$$\tau_{(r,n)}(D_1) = D_2,$$

where $k = Fr(D)$ and $F \triangleq k(w_1, w_2, \dots, w_r)$ is assumed to be contained in the intersection $Fr(D_1) \cap Fr(D_2)$.

Let X and Y be two integral varieties and let $\varphi : X \rightarrow Y$ be a surjective morphism. Fixed an algebraically closed closure Ω of the function field $k(X)$.

Definition 3.4. (see [3]) A reduced affine covering \mathcal{C}_X of X with values in Ω is said to be **quasi-galois closed** over Y by φ if the below condition is satisfied:

There exists a local chart $(U'_\alpha, \phi'_\alpha; A'_\alpha) \in \mathcal{C}_X$ such that $U'_\alpha \subseteq \varphi^{-1}(V_\alpha)$ for any $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$, for any affine open set V_α in Y with $U_\alpha \subseteq \varphi^{-1}(V_\alpha)$, and for any conjugation A'_α of A_α over B_α , where B_α is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in the function field $k(Y)$.

We have the following lemmas for the criterion of quas-galois closed.

Lemma 3.5. (c.f. [3]) *Assume that $k(Y)$ is contained in Ω . Then X is quasi-galois closed over Y if there is an affine patching \mathcal{C}_X of X with values in Ω such that*

- either \mathcal{C}_X is quasi-galois closed over Y ,
- or A_α has only one conjugate over B_α for any $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ and for any affine open set V_α in Y with $U_\alpha \subseteq \varphi^{-1}(V_\alpha)$, where B_α is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in the function field $k(Y)$.

Proof. It is immediate from definition. \square

We have the lemma below for the existence of quasi-galois closed.

Lemma 3.6. (see [3]) *Let K be a finitely generated extensions of a number field and let Y be an arithmetic variety with $K = k(Y)$. Fixed any finitely generated extensions L of K such that L is Galois over K .*

Then there exists an arithmetic variety X and a surjective morphism $f : X \rightarrow Y$ of finite type such that

- $L = k(X)$;
- the morphism f is affine;
- X is a quasi-galois closed over Y by f .

We have the lemma below for the property of quasi-galois closed.

Lemma 3.7. (see [2]) *Let X and Y be two arithmetic varieties. Assume that X is quasi-galois closed over Y by a surjective morphism ϕ of finite type.*

Then the function field $k(X)$ is canonically a Galois extension of $k(Y)$ and there is a group isomorphism

$$\text{Aut}(X/Y) \cong \text{Gal}(k(X)/k(Y)).$$

Moreover, let $\dim X = \dim Y$. Then ϕ is finite and X is a pseudo-galois cover of Y in the sense of Suslin-Voevodsky.

At last we have the useful lemma below.

Lemma 3.8. (c.f. [4]) *Let X be an integral variety. Then there is an integral variety Z satisfying the conditions:*

- $k(X) = k(Z) \subseteq \Omega$;
- $X \cong Z$ are isomorphic;
- Z has a reduced affine covering with values in Ω .

Proof. Trivial. \square

3.2. Recalling the construction for a geometric model. (see [3, 4]) Let Y be an arithmetic variety. Let L be a Galois extension of the function field $K = k(Y)$. In the following we will proceed in several steps to construct a scheme X and a surjective morphism $f : X \rightarrow Y$ such that $L = k(X)$ and X is quasi-galois closed over Y by f .

By Lemma 3.8, without loss of generality, assume that there is a reduced affine covering \mathcal{C}_Y of the scheme Y with values in Ω .

Step 1. Fixed a set Δ of the generators L over K . That is, $L = K(\Delta)$ and $\Delta \subseteq L \setminus K$. Put $G = \text{Gal}(L/K)$.

Step 2. Take any local chart $(V, \psi_V, B_V) \in \mathcal{C}_Y$. Define A_V to be the subring of L generated over B_V by the set

$$\Delta_V = \{\sigma(x) \in L : \sigma \in G, x \in \Delta\}.$$

We have

$$A_V = B_V[\Delta_V].$$

Set

$$i_V : B_V \rightarrow A_V$$

to be the inclusion.

Step 3. Define

$$\Sigma = \coprod_{(V, \psi_V, B_V) \in \mathcal{C}_Y} \text{Spec}(A_V)$$

to be the disjoint union and define

$$\pi_Y : \Sigma \rightarrow Y$$

to be the projection.

Then Σ is a topological space, where the topology τ_Σ on Σ is naturally determined by the Zariski topologies on all $\text{Spec}(A_V)$.

Step 4. Define an equivalence relation R_Σ in Σ in such a manner:

For any $x_1, x_2 \in \Sigma$, we say $x_1 \sim x_2$ if and only if $j_{x_1} = j_{x_2}$ holds in L . Here j_x denotes the corresponding prime ideal of A_V to a point $x \in \text{Spec}(A_V)$ (see [7]).

Define

$$X = \Sigma / \sim.$$

Let

$$\pi_X : \Sigma \rightarrow X$$

be the projection.

Step 5. Define a map

$$f : X \rightarrow Y$$

by

$$\pi_X(z) \longmapsto \pi_Y(z)$$

for each $z \in \Sigma$.

Step 6. Put

$$\mathcal{C}_X = \{(U_V, \varphi_V, A_V)\}_{(V, \psi_V, B_V) \in \mathcal{C}_Y}$$

where $U_V = \pi_Y^{-1}(V)$ and $\varphi_V : U_V \rightarrow \text{Spec}(A_V)$ is the identity map for each $(V, \psi_V, B_V) \in \mathcal{C}_Y$.

Define the scheme (X, \mathcal{O}_X) to be obtained by gluing the affine schemes $\text{Spec}(A_V)$ for all local charts $(V, \psi_V, B_V) \in \mathcal{C}_Y$ with respect to the equivalence relation R_Σ (see [7, 9]).

Then (X, \mathcal{O}_X) is the desired scheme and $f : X \rightarrow Y$ is the desired morphism of schemes. This completes the construction.

3.3. Any two qc varieties have a common qc cover. Let X be an arithmetic variety and let Ω be an algebraically closed field containing $k(X)$. We need the following lemma in order to prove that $X_{qc}[\Omega]$ is a directed set.

Lemma 3.9. *Take any $Z_1, Z_2 \in X_{qc}[\Omega]$. There is a third $Z_3 \in X_{qc}[\Omega]$ such that Z_3 is quasi-galois closed over Z_1 and Z_2 , respectively.*

Proof. Let $f_1 : Z_1 \rightarrow X$ and $Z_2 \rightarrow X$ be two surjective morphisms of finite types such that Z_1/X and Z_2/X are quasi-galois closed. It is seen that f_1 and f_2 are both affine by *Definition 1.1* and *Lemma 3.6*. And it is seen that the function fields $k(Z_1)/k(X)$ and $k(Z_2)/k(X)$ are both Galois from *Lemma 3.7*.

Without loss of generality, by *Lemma 3.8*, assume that X has a reduced affine covering \mathcal{C}_X with values in Ω .

Fixed any affine chart V contained in \mathcal{C}_X . Put

$$\Delta = \Delta_1 \cup \Delta_2$$

and

$$G = \text{Gal}(k(Z_1) \cdot k(Z_2)/k(X)).$$

Here, Δ_j is the set of generators of the ring $\mathcal{O}_{Z_j}(f_j^{-1}(V))$ over $\mathcal{O}_X(V)$ for $j = 1, 2$.

Repeat the construction in the previous subsection §3.2 for the set Δ and the Galois group G . Then we obtain an arithmetic variety Z_3 .

By *Lemma 3.5* it is clear that Z_3/X , Z_3/Z_1 , and Z_3/Z_2 are all quasi-galois closed. Hence, $Z_3 \in X_{qc}[\Omega]$. \square

Lemma 3.10. *Let $Z_1, Z_2, Z_3 \in X_{qc}[\Omega]$ be such that Z_1/Z_2 and Z_2/Z_3 are both quasi-galois closed. Then Z_1/Z_3 is quasi-galois closed.*

Proof. Let $f_1 : Z_1 \rightarrow Z_2$ and $f_2 : Z_2 \rightarrow Z_3$ be surjective morphisms of finite types by which Z_1/Z_2 and Z_2/Z_3 are both quasi-galois closed. Put $f_3 = f_2 \circ f_1$.

Then $f_3 : Z_1 \rightarrow Z_3$ is a surjective morphism of finite type. It is seen that f_3 is affine since f_1 and f_2 are both affine. By *Lemma 3.5* it follows that Z_1/Z_3 is quasi-galois closed by f_3 . \square

3.4. A property for qc integral varieties. There is a result for integral varieties which is similar to arithmetic varieties (see Lemma 3.7 above or [2]).

Lemma 3.11. *Let $f : X \rightarrow Y$ be a surjective morphism of integral varieties. Suppose that X/Y is quasi-galois closed by f and that $k(X)$ is canonically a Galois extension over $k(Y)$. Then f is affine and there is a group isomorphism*

$$\text{Aut}(X/Y) \cong \text{Gal}(k(X)/k(Y)).$$

Proof. By Lemma 3.8, without loss of generality, assume that X and Y have reduced affine coverings \mathcal{C}_X and \mathcal{C}_Y with values in $\Omega = k(X)^{al}$, respectively.

Let

$$\Delta \subseteq k(X) \setminus k(Y)$$

be the set of generators of $k(X)$ over $k(Y)$. Put

$$G = \text{Gal}(k(X)/k(Y)).$$

Repeat the construction for a geometric model in §3.2, we obtain an integral variety X' such that A_α has only one conjugate over B_α for any $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ and for any affine open set V_α in Y with $U_\alpha \subseteq \varphi^{-1}(V_\alpha)$, where $B_\alpha = \mathcal{O}_Y(V_\alpha)$.

By Lemma 3.5 it is seen that X'/Y is quasi-galois closed. It follows that $X' = X$ from Remark 1.2. Hence, f must be affine.

Now define a mapping

$$t : \text{Aut}(X/Y) \longrightarrow \text{Gal}(k(X)/k(Y))$$

by

$$\sigma = (\sigma, \sigma^\sharp) \longmapsto t(\sigma) = \langle \sigma, \sigma^{\sharp-1} \rangle$$

where $\langle \sigma, \sigma^{\sharp-1} \rangle$ is the map of $k(X)$ into $k(X)$ given by

$$(U, f) \in \mathcal{O}_X(U) \subseteq k(X) \longmapsto (\sigma(U), \sigma^{\sharp-1}(f)) \in \mathcal{O}_X(\sigma(U)) \subseteq k(X)$$

for any $f \in \mathcal{O}_X|_U(U)$, where U runs through all open sets in X .

It is easily seen that t is well-defined.

As we have done for the proof of the main theorem in [2], in the following we will proceed in several steps to prove that t is a group isomorphism.

Step 1. Prove that t is injective. Take any $\sigma, \sigma' \in \text{Aut}(X/Y)$ such that $t(\sigma) = t(\sigma')$. We have

$$(\sigma(U), \sigma^{\sharp-1}(f)) = (\sigma'(U), \sigma'^{\sharp-1}(f))$$

for any $(U, f) \in k(X)$. In particular, for any $f \in \mathcal{O}_X(U_0)$ we have

$$(\sigma(U_0), \sigma^{\sharp-1}(f)) = (\sigma'(U_0), \sigma'^{\sharp-1}(f))$$

where U_0 is an affine open subset of X such that $\sigma(U_0)$ and $\sigma'(U_0)$ are both contained in $\sigma(U) \cap \sigma'(U)$.

It is seen that

$$\sigma|_{U_0} = \sigma'|_{U_0}$$

holds as isomorphisms of schemes. As U_0 is dense in X , we have

$$\sigma = \sigma|_{\overline{U_0}} = \sigma'|_{\overline{U_0}} = \sigma'$$

on the whole of X ; then

$$\sigma(U) = \sigma'(U);$$

it follows that

$$\sigma = \sigma'$$

holds.

Step 2. Prove that t is surjective. Fixed any element ρ of the group $\text{Gal}(k(X)/k(Y))$.

As $k(X) = \{(U_f, f) : f \in \mathcal{O}_X(U_f) \text{ and } U_f \subseteq X \text{ is open}\}$, we have

$$\rho : (U_f, f) \in k(X) \longmapsto (U_{\rho(f)}, \rho(f)) \in k(X),$$

where U_f and $U_{\rho(f)}$ are open sets in X , f is contained in $\mathcal{O}_X(U_f)$, and $\rho(f)$ is contained in $\mathcal{O}_X(U_{\rho(f)})$.

In fact, we prove that each element of $\text{Gal}(k(X)/k(Y))$ give us a unique element of $\text{Aut}(X/Y)$:

Fixed any affine open set V of Y . It is easily seen that for each affine open set $U \subseteq \phi^{-1}(V)$ there is an affine open set U_ρ in X such that ρ determines an isomorphism λ_U between affine schemes $(U, \mathcal{O}_X|_U)$ and $(U_\rho, \mathcal{O}_X|_{U_\rho})$.

Take any affine open sets $V \subseteq Y$. It is seen that

$$\lambda_U|_{U \cap U'} = \lambda_{U'}|_{U \cap U'}$$

holds as morphisms of schemes for any affine open sets $U, U' \subseteq \phi^{-1}(V)$.

By gluing λ_U along all such affine open subsets U , we have an automorphism λ of the scheme X such that $\lambda|_U = \lambda_U$ for any affine open set U in X . It is easily seen that $t(\lambda) = \rho$. Hence, t is surjective. This completes the proof. \square

Remark 3.12. Let X and Y be integral varieties and let X be quasi-galois closed over Y by a surjective morphism ϕ . Then there is a natural isomorphism

$$\mathcal{O}_Y \cong \phi_*(\mathcal{O}_X)^{\text{Aut}(X/Y)}$$

where $(\mathcal{O}_X)^{Aut(X/Y)}(U)$ denotes the invariant subring of $\mathcal{O}_X(U)$ under the natural action of $Aut(X/Y)$ for any open subset U of X .

3.5. A universal cover for the qc fundamental group. Let X be an arithmetic variety and let Ω be an algebraically closed field such that $k(X)^{al} \subseteq \Omega$. Without loss of generality, assume that X has a reduced affine covering with values in $k(X)^{al}$.

Put

$$G = Gal(\Omega/k(X))$$

and take a set

$$\Delta \subseteq \Omega \setminus k(X)$$

of generators of Ω over $k(X)$. Repeat the construction in §3.2, we have an integral variety X_Ω such as in the following lemma.

Lemma 3.13. *Let X be an arithmetic variety and let Ω be an algebraically closed field containing $k(X)^{al}$. Then there is an integral variety X_Ω and a surjective morphism $f_\Omega : X_\Omega \rightarrow X$ satisfying the conditions:*

- $k(X_\Omega) = \Omega$;
- f_Ω is affine;
- $k(X_\Omega)$ is Galois over $k(X)$;
- X_Ω/X is quasi-galois closed by f_Ω .

Such an integral variety X_Ω is called a **geometric model** for the qc fundamental group $\pi_1^{qc}(X; \Omega)$ or a **universal cover** over X for the group $\pi_1^{qc}(X; \Omega)$.

3.6. A universal cover for the étale fundamental group. Let X be an arithmetic variety and let s be a geometric point of X over $k(X)^{al}$. Without loss of generality, assume that X has a reduced affine covering with values in $k(X)^{al}$.

Put

$$\begin{aligned} \Omega_{et} &= k(X)^{un}; \\ G &= Gal(\Omega_{et}/k(X)). \end{aligned}$$

Take a set

$$\Delta \subseteq \Omega_{et} \setminus k(X)$$

of generators of Ω_{et} over $k(X)$. Repeat the construction in §3.2, we have an integral variety $X_{\Omega_{et}}$ such as in the following lemma.

Lemma 3.14. *Let X be an arithmetic variety and let s be a geometric point of X over $k(X)^{al}$. Then there is an integral variety $X_{\Omega_{et}}$ and a surjective morphism $f_{\Omega_{et}} : X_{\Omega_{et}} \rightarrow X$ satisfying the conditions:*

- $k(X_{\Omega_{et}}) = \Omega_{et}$;

- $f_{\Omega_{et}}$ is affine;
- $k(X_{\Omega_{et}})$ is Galois over $k(X)$;
- $X_{\Omega_{et}}/X$ is quasi-galois closed by $f_{\Omega_{et}}$.

Such an integral variety $X_{\Omega_{et}}$ is called a **geometric model** for the étale fundamental group $\pi_1^{et}(X; s)$ or a **universal cover** over X for the group $\pi_1^{et}(X; s)$.

3.7. Proof of the main theorem. Now we can prove the main theorem of the paper.

Proof. (Proof of Theorem 2.2) We will proceed in several steps.

Step 1. By *Lemma 3.13* we have

$$\begin{aligned} Gal(\Omega/k(X)) &\cong \varprojlim_{Z \in X_{qc}[\Omega]} Gal(k(Z)/k(X)) \\ &\cong \varprojlim_{Z \in X_{qc}[\Omega]} Aut(Z/X) \\ &= \pi_1^{qc}(X; \Omega). \end{aligned}$$

Step 2. Let $X_{et}[s]$ be the set of finite étale Galois covers of X with respect to the geometric point s (see). For any $Z_1, Z_2 \in X_{et}[s]$ we say

$$Z_1 \leq Z_2$$

if and only if Z_2 is a finite étale Galois cover of Z_1 . Then $X_{et}[s]$ is a partially ordered set. Put

$$X_{qc}[\Omega; s] = X_{qc}[\Omega] \cap X_{et}[s].$$

Let $Z_1, Z_2 \in X_{qc}[\Omega; s]$. From *Lemma 3.5* it is easily seen that Z_2/Z_1 is a finite étale Galois cover if and only if Z_2/Z_1 is quasi-galois closed. Hence, the two partial orders \leq in $X_{qc}[\Omega; s]$ coincide with each other.

It follows that $X_{qc}[\Omega; s]$ and $X_{et}[s]$ are cofinal directed sets according to the construction in § 3.2.

Step 3. By the construction in § 3.2 again, for the universal covers we have

$$X_{\Omega} = \varinjlim_{Z \in X_{qc}[\Omega]} Z$$

and

$$X_{\Omega_{et}} = \varinjlim_{Z \in X_{et}[s]} Z$$

as direct limits.

From *Lemma 3.14* we have

$$\begin{aligned} Gal(\Omega_{et}/k(X)) &\cong \varprojlim_{Z \in X_{et}[s]} Gal(k(Z)/k(X)) \\ &\cong \varprojlim_{Z \in X_{et}[s]} Aut(Z/X) \\ &\cong \pi_1^{et}(X; s). \end{aligned}$$

On the other hand, by *Step 2* we have

$$\begin{aligned} & \lim_{\leftarrow} \text{Gal}(k(Z)/k(X)) \\ & \cong \text{Gal}(\Omega_{et}/k(X)) \\ & \cong \lim_{\leftarrow} \text{Gal}(k(Z)/k(X)). \end{aligned}$$

Hence,

$$\pi_1^{et}(X; s) \cong \text{Gal}(\Omega_{et}/k(X)).$$

It follows that $\pi_1^{et}(X; s)$ is a subgroup of $\pi_1^{qc}(X; \Omega)$. It is evident that $\pi_1^{et}(X; s)$ is normal in the group $\pi_1^{qc}(X; \Omega)$ since $k(X_{\Omega_{et}})$ is a subfield of $k(X_{\Omega})$ and $k(X_{\Omega_{et}})$ is Galois over $k(X)$.

This completes the proof. \square

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